

# GENERALIZED ALEXANDER POLYNOMIAL INVARIANTS<sup>1</sup>

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**Abstract** We propose an algorithm which allows to derive the generalized Alexander polynomial invariants of knots and links with the help of the  $q, p$ –numbers, appearing in bosonic two-parameter quantum algebra. These polynomials turn into HOMFLY ones by applying special parametrization. The Jones polynomials can be also obtained by using this algorithm.

**Keywords:** knots and links; skein relationship; recurrence relation;  $q$ –numbers;  $q, p$ –numbers; polynomial invariants of knots and links; Alexander polynomials; generalized Alexander polynomials; HOMFLY polynomials; Jones polynomials.

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# 1 Introduction

The aim of this paper is to generalize one-parameter Alexander polynomial invariants, one of the main characteristics of knots and links, to two-parameter generalized Alexander polynomial invariants.

First, we recall some basic notions of the knot theory. Applying to an initial link (knot)  $L_+$  so called “surgery” operation - elimination of a crossing - we obtain a simpler link/knot  $L_O$ . Applying to the same initial link (knot)  $L_+$  another “surgery” operation - switching of a crossing - we obtain another simpler link/knot  $L_-$ .

It is postulated:

- 1) every knot and link is described by the definite polynomial;
- 2) three concrete polynomials, namely  $P_{L_+}(t)$ ,  $P_{L_O}(t)$ ,  $P_{L_-}(t)$  are connected with the help of the following (geometro-algebraic) recurrence relation, which is called the skein relationship:

$$P_{L_+}(t) = l_1 P_{L_O}(t) + l_2 P_{L_-}(t) \quad (1)$$

where  $l_1, l_2$  are coefficients;

- 3) the normalization condition for the unknot:

$$P_{unknot} = 1. \quad (2)$$

Applying the operation of elimination for torus knots and links  $L_{n,2}$  turns it into  $L_{n-1,2}$ , and the switching operation turns it into  $L_{n-2,2}$ , where  $n$  is a positive integer number. From these considerations and from (1) it follows the following recurrence relation :

$$P_{L_{n+1,2}}(t) = l_1 P_{L_{n,2}}(t) + l_2 P_{L_{n-1,2}}(t),$$

or in the simpler notations:

$$P_{n+1,2}(t) = l_1 P_{n,2}(t) + l_2 P_{n-1,2}(t). \quad (3)$$

Thus, the form of the recurrence relation (3) for torus knots and links  $L_{n,2}$  coincides with the form of the skein relationship (1).

Recurrence relation only for torus knots  $T(2m+1, 2)$  (or only for torus links  $L(2m, 2)$ ), where  $m = 0, 1, 2, \dots$  looks as follows:

$$P_{n+2,2}(t) = k_1 P_{n,2}(t) + k_2 P_{n-2,2}(t), \quad (4)$$

where

$$k_1 = l_1^2 + 2l_2, \quad k_2 = -l_2^2. \quad (5)$$

We also have

$$P_{1,2} = 1, \quad P_{3,2} = k_1 + k_2. \quad (6)$$

## 2 Alexander polynomials

The Alexander polynomials  $\Delta(t)$  of knots and links [1] can be defined by the skein relationship

$$\Delta_+(t) - \Delta_-(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_O(t), \quad \Delta_{unknot} = 1. \quad (7)$$

From (7) (in analogy to (3)) it follows the following recurrence relation for torus knots and links  $L_{n,2}(t)$ :

$$\Delta_{n+1,2}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{n,2}(t) + \Delta_{n-1,2}(t). \quad (8)$$

From (8) (in analogy to (4)) one obtains the recurrence relation only for torus knots  $T(2m+1, 2)$  (or for torus links  $L(2m, 2)$ )

$$\Delta_{n+2,2}(t) = (t + t^{-1})\Delta_{n,2}(t) - \Delta_{n-2,2}(t). \quad (9)$$

The Alexander polynomials of torus knots  $T(n, 2)$  can be expressed through  $q$ -numbers characteristic to Biedenharn-Macfarlane quantum bosonic oscillator. The bosonic  $q$ -number corresponding to an integer  $n$  is defined as [2, 3]

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (10)$$

where  $q$  is a parameter. Some of the  $q$ -numbers are:

$$[1]_q = 1, \quad [2]_q = q + q^{-1}, \quad [3]_q = q^2 + 1 + q^{-2}, \quad [4]_q = q^3 + q + q^{-1} + q^{-3}, \dots$$

The recurrence relation for (10) looks as

$$[n+1]_q = (q + q^{-1})[n]_q - [n-1]_q. \quad (11)$$

It was found that [4, 5]:

$$\Delta_{2m+1,2}(t) = [m+1]_t - [m]_t, \quad t \equiv q, \quad (12)$$

or, since  $n = 2m + 1$ , as

$$\Delta_{n,2}(t) = \left[ \frac{n+1}{2} \right]_t - \left[ \frac{n-1}{2} \right]_t. \quad (13)$$

In the following section we generalize these results with the help of  $q$ -numbers.

### 3 Algorithm of obtaining of Alexander polynomials from bosonic $q$ -numbers

Analyzing the results of previous sections we can formulate an algorithm of obtaining of the Alexander skein relationship (7). Afterwards this procedure will be used for obtaining another skein relations.

First step: we introduce polynomials  $A_{n,2}(q)$ , which refer to torus knots  $T(2m+1, 2)$ , satisfying following recurrence relation (repeating (11)):

$$A_{n+2,2}(q) = (q + q^{-1})A_{n,2}(q) - A_{n-2,2}(q). \quad (14)$$

According to (6):

$$A_{1,2}(q) = 1, \quad A_{3,2}(q) = q - 1 + q^{-1}. \quad (15)$$

Second step: we formulate full recurrence relation for all polynomials  $A_{n,2}(q)$  and, thus, find corresponding skein relationship. From (14) we have  $k_1 = q + q^{-1}$ ,  $k_2 = -1$ . Because of (5), we find

$$l_1 = q^{\frac{1}{2}} + q^{-\frac{1}{2}}, \quad l_2 = 1. \quad (16)$$

Therefore

$$A_{n+1,2}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})A_{n,2}(q) + A_{n-1,2}(q). \quad (17)$$

From (17) (in analogy with (1) and (3)) we obtain the following skein relationship:

$$A_+(q) - A_-(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})A_O(q). \quad (18)$$

Third step: we find an expression for torus knots  $A_{2m+1,2}(q)$ . In analogy with (19), we put

$$A_{2m+1,2}(q) = a_1(q)[m+1]_q - a_2(q)[m]_t, \quad t \equiv q, \quad (19)$$

Using (10), (15) and (19), we find  $a_1(q) = 1$ ,  $a_2(q) = 1$ . Therefore,

$$A_{2m+1,2}(q) = [m+1]_q - [m]_q. \quad (20)$$

In general, we described three-step procedure of obtaining of: 1) skein relationship of knots and links, and 2) expression for polynomial invariants of torus knots  $T(2m+1, 2)$ , from structural functions of bosonic deformed oscillators. In particular, we obtained the formulas (18), (28), which coincides with those for the Alexander polynomial invariants (7), (12) (if  $q \equiv t$ ).

## 4 Generalized Alexander polynomials $A(q, p)$ from $q, p$ -numbers

In this section we use the proposed three-step algorithm to obtain the generalized Alexander polynomials  $A(q, p)$  from  $q, p$ -numbers, which reduce to the Alexander polynomials if  $p = q^{-1}$ .

The  $q, p$ -number corresponding to integer number  $n$  is introduced as [6]

$$[n]_{q,p} = \frac{q^n - p^n}{q - p}, \quad (21)$$

where  $q, p$  are some complex parameters. If  $p = q^{-1}$ , then  $[n]_{q,p} = [n]_q$ . Here are some of the  $q, p$ -numbers:

$$[1]_{q,p}=1, \quad [2]_{q,p}=q+p, \quad [3]_{q,p}=q^2+qp+p^2, \quad [4]_{q,p}=q^3+q^2p+qp^2+p^3, \dots$$

The recurrence relation for  $q, p$ -numbers is

$$[n+1]_{q,p} = (q+p)[n]_{q,p} - qp[n-1]_{q,p}. \quad (22)$$

First, in analogy with previous section, on the base of (22) we introduce polynomials  $A_{n,2}(q, p)$ , which generalize the Alexander polynomials:

$$A_{n+2,2}(q, p) = (q+p)A_{n,2}(q, p) - qpA_{n-2,2}(q, p). \quad (23)$$

Thus from normalization condition and (6)

$$A_{1,2}(q, p) = 1, \quad A_{3,2}(q, p) = q - qp + p. \quad (24)$$

Second, from (23) it also follows

$$k_1 = l_1^2 + 2l_2 = q + p, \quad k_2 = -l_2^2 = -qp.$$

From here one finds

$$l_2 = q^{\frac{1}{2}}p^{\frac{1}{2}}, \quad l_1 = q^{\frac{1}{2}} - p^{\frac{1}{2}},$$

which leads to the generalized Alexander skein relationship [7]:

$$A_+(q, p) = (q^{\frac{1}{2}} - p^{\frac{1}{2}})A_O(q, p) + q^{\frac{1}{2}}p^{\frac{1}{2}}A_-(q, p). \quad (25)$$

Formula (25) can be written in the form

$$q^{-\frac{1}{4}}p^{-\frac{1}{4}}A_+(q, p) - q^{\frac{1}{4}}p^{\frac{1}{4}}A_-(q, p) = (q^{\frac{1}{4}}p^{-\frac{1}{4}} - q^{-\frac{1}{4}}p^{\frac{1}{4}})A_O(q, p) \quad (26)$$

By putting  $p = q^{-1}$ , the generalized Alexander skein relationship turns into the Alexander skein relationship (7).

Third, we take

$$A_{2m+1,2}(q, p) = a_1(q, p)[m+1]_{q,p} - a_2(q, p)[m]_{q,p}. \quad (27)$$

From (24) we have  $a_1(q, p) = 1$ ,  $a_2(q, p) = qp$ . Therefore,

$$A_{2m+1,2}(q, p) = [m+1]_{q,p} - qp[m]_{q,p}. \quad (28)$$

## 5 Generalized Alexander polynomials and HOMFLY polynomials

The HOMFLY polynomial invariants [8] are described by the skein relationship:

$$a^{-1}H_+(a, z) - aH_-(a, z) = zH_O(a, z). \quad (29)$$

Comparing (26) with the HOMFLY skein relationship (29) we obtain

$$a = q^{\frac{1}{4}}p^{\frac{1}{4}}, \quad z = q^{\frac{1}{4}}p^{-\frac{1}{4}} - q^{-\frac{1}{4}}p^{\frac{1}{4}}. \quad (30)$$

Substituting this result into (29), one obtains the generalized Alexander skein relationship (26).

## 6 Generalized Alexander polynomials and Jones polynomials

The Jones polynomial invariants [9] can be defined as

$$t^{-1}V_+(t) - tV_-(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_O(t). \quad (31)$$

Comparing (26) with the Jones skein relationship (31), we find that substitution

$$q = t^3, \quad p = t \quad (32)$$

reduces the generalized Alexander polynomials to Jones ones.

According to results of Section 3, the Jones skein relationship (31) can be obtained with the help of the proposed three-step algorithm from  $q$ -numbers defined as

$$[n]_{q^3, q} = \frac{q^{3n} - q^n}{q^3 - q}. \quad (33)$$

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